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MULTIPLICATIVE UTILITY FUNCTIONS

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by

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13 ABSTRACT

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FOREWORD

The Operations Research Center at the Massachusetts Institute of
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Abstract

This paper presents necessary and sufficient conditions for a multi-attributed utility function to be either multiplicative or additive. It is shown that the additive form is a limiting case of the multiplicative form. The number of requisite assumptions to imply the main result is equal to the number of attributes. Because the assumptions involve only trade-offs between two attributes at a time or lotteries over one attribute, it is reasonable to expect that decision makers can ascertain if the assumptions are appropriate for their specific problem. Procedures are given for verifying the assumptions and assessing the resulting utility functions. The paper concludes with a discussion of a recent application of the results to a six-attribute problem relating to the development of Mexico City's airport facilities.

1. INTRODUCTION

Most complex problems involve multiple objectives. Thus, analytical work on such problems requires that one obtain an objective function involving multiple measures of effectiveness (attributes) which indicate the degree to which these objectives are met. Such an objective function specifies a preference ranking of consequences and allows one to identify the trade-offs between various levels of the different attributes. In a risk-free environment, one should choose the alternative course of action which maximizes (or minimizes) the objective function.

However, most real decision problems involve uncertainties--and these uncertainties need to be either formally or informally considered in analyzing a problem. If one chooses to do this formally, it is necessary to specify an objective function with special characteristics in order to make the analytics for solving the problem tractable. For this reason, it would be nice to be able to use the expected value of the objective function as a guide to identify the best alternatives. This is appropriate given one accepts the axiom's of utility theory specified by von Neumann and Morgenstern[17] and the objective function is a utility function. The utility function still provides one with the necessary information to rank consequences and identify trade-offs between attributes, but it also follows from the aforementioned axioms that one should choose the alternative that maximizes the expected utility.

The utility concept is theoretically sound, and the mathematical details are not involved. However, the difficulty comes when one tries to

specify reasonable procedures for obtaining multiattributed utility functions. The general approach followed by many people has been to make assumptions about preferences and then derive the functional form(s) of the utility function satisfying these assumptions. For a real problem, if the assumptions are verified, the functional form can be used to simplify the requisite assessments needed to specify the utility function. Often these assumptions are so involved that it is unreasonable to expect a decision maker to ascertain whether or not they might be appropriate for a specific problem.

In this paper we state necessary and sufficient conditions to imply that a multiattributed utility function is either multiplicative or additive. The number of conditions required increase linearly with the number of these attributes. None of the conditions require the decision maker to consider trade-offs between more than two attributes simultaneously or to consider lotteries over more than one attribute. Furthermore, subject to the assumptions, the assessments needed to completely specify the n-attribute utility function are n one-attribute utility functions and n scaling constants.

In Section 2, after defining the necessary notation, we state the main result, which is proven in the appendix. The manner in which one would assess the multiplicative or additive utility function is discussed in Section 3. Sections 4 and 5 present related results of others and outline the use of our result in one real-world decision problem.

2. THE MAIN RESULT

Let $X \equiv X_1 \times X_2 \times \cdots \times X_n$ be a consequence space, where X_i is the ith attribute. A specific consequence will be designated by $\underline{x}$ or $(x_1, x_2, \ldots, x_n)$. We are interested in assessing the utility function over X, which will be denoted by $u(x_1, x_2, \ldots, x_n)$ or $u(\underline{x})$. Finally, $X_{\overline{ij}}$ will mean $X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_{j-1} \times X_{j+1} \times \cdots \times X_n$ and $x_{\overline{ij}}$ will be a member of $X_{\overline{ij}}$. Similarly, the notation $X_{\overline{i}} \equiv X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$, and $x_{\overline{i}}$ is a member of $X_{\overline{i}}$. In all that follows, each X_i may be a scalar attribute (i.e., take on scalar values) or a cartesian product of scalar attributes.

The Basic Assumptions

The main assumptions which we use concern the concepts preferential independence and utility independence. We will say $X_i \times X_j$ is <u>preferentially independent</u> of $X_{\overline{ij}}$ if one's preference order for consequences $(x_i, x_j, x_{\overline{ij}})$, with $x_{\overline{ij}}$ held fixed does not depend on the fixed amount $x_{\overline{ij}}$. This is equivalent to assuming trade-offs under certainty between various amounts of X_i and X_j do not depend on $X_{\overline{ij}}$. The preferential independence assumption implies that the indifference curves over $X_i \times X_j$ are the same regardless of the value of $X_{\overline{ij}}$.

In a similar fashion, we say X_i is <u>utility independent</u> of $X_{\overline{i}}$ if one's preference order over lotteries on X_i, written $(\tilde{x}_i, x_{\overline{i}})$, with $X_{\overline{i}}$ held fixed does not depend on the fixed amount $x_{\overline{i}}$. This implies the conditional utility

function over X_i, given $X_{\bar{i}}$ is fixed at any value, will be a positive linear transformation of the conditional utility function over X_i, given $X_{\bar{i}}$ is fixed at any other value.

With these ideas, we can state our main result.

THEOREM 1. Let $X \equiv X_1 \times X_2 \times \cdots \times X_n$, $n \geq 3$. If for some X_i, $X_i \times X_j$ is preferentially independent of $X_{\overline{ij}}$ for all $j \neq i$ and X_i is utility independent of $X_{\bar{i}}$, then either

$$u(\underline{x}) = \sum_{r=1}^{n} k_r u_r(x_r),\tag{1}$$

or

$$1 + k\, u(\underline{x}) = \prod_{r=1}^{n} [1 + k\, k_r\, u_r(x_r)],\tag{2}$$

where u and the u_r are utility functions scaled from zero to one, the k_r are scaling constants with $0 < k_r < 1$, and $k > -1$ is a scaling constant.

Equation (1) is the additive utility function. When k is positive in (2), then $u^*(\underline{x}) \equiv 1 + k\, u(\underline{x})$ and $u_r^*(x_r) \equiv 1 + k\, k_r u_r(x_r)$ are utility functions over the appropriate domains and $0 < k_r < 1$, and $k > -1$ is a scaling constant

$$u^*(\underline{x}) = \prod_{r=1}^{n} u_r^*(x_r) \ .$$

When k is negative, note that $u^*(x) \equiv -[k\,u(x) + 1]$ and $u_r^*(x_r) \equiv -[1 + k\, k_r u_r(x_r)]$ are utility functions over X and X_r, respectively so

$$-u^*(\underline{x}) = (-1)^n \prod_{r=1}^{n} u_r^*(x_r).$$

Hence we can refer to form (2) as a multiplicative utility function.

The proof of Theorem 1 requires the following three results:

Lemma 1. If X_i is utility independent of $X_{\bar{i}}$ and if

$X_i \times X_j$ is preferentially independent of

$X_{\overline{ij}}$, then $X_i \times X_j$ is utility independent of

$X_{\overline{ij}}$.

Lemma 2. If $X_i \times X_j$ is utility independent of $X_{\overline{ij}}$ and

$X_i \times X_k$ are utility independent of $X_{\overline{ik}}$,

then $X_i \times X_j \times X_k$ is utility independent of

$X_{\overline{ijk}}$.

Lemma 3. If $X_{\bar{i}}$ is utility independent of X_i, for n-1

of the i = 1, 2, ..., n, and n $\geq$ 3, then

either equation (1) or (2) is valid.

The detailed proofs of these results are found in the appendix.

Proof of Theorem 1

The proof is different for the case when n $\geq$ 4 and the case where

n = 3. When n $\geq$ 4, given the assumptions stated in the theorem, it follows

from Lemma 1 that $X_i \times X_j$ is utility independent of $X_{\overline{ij}}$ for all j. Lemma 2

says that when overlapping sets of attributes are utility independent

of their complementary sets, their union is utility independent of its complement. Thus, by repeated use of Lemma 2, we can conclude that $X_{\bar{j}}$ is utility independent of X_j for all $j \neq i$. The final result then follows directly from Lemma 3.

In the case where $n = 3$, the proof of Theorem 1 follows directly from invoking Lemmas 1 and 3.

For the case of two attributes, since the preferential independence assumptions do not apply, our theorem is not appropriate. In this case, it is proven in Keeney[10] that the utility function is either additive or multiplicative, corresponding to (1) and (2), if and only if X_1 is utility independent of X_2 and X_2 is utility independent of X_1.

Given the conditions of Theorem 1 hold, it is important to know whether the utility function is additive or multiplicative. With regards to this, we state without proof the following

COROLLARY. Suppose the requisite assumptions of Theorem 1 obtain. In addition, if for any one $x_{\overline{ij}}$, we can find two different amounts of X_i, call them x_i and x_i', and two different amounts of X_j, call them x_j and x_j', such that the decision maker is indifferent between a lottery yielding either $(x_i, x_j, x_{\overline{ij}})$ or $(x_i', x_j', x_{\overline{ij}})$ with equal probability or a lottery yielding either $(x_i, x_j', x_{\overline{ij}})$ or $(x_i', x_j, x_{\overline{ij}})$ with equal probability, then the utility function must be additive. If he has a preference between the two lotteries, then the utility function will be multiplicative.

If the indifference or preference condition for the lotteries holds for one $x_{\overline{ij}}$, it can be shown to hold for all $x_{\overline{ij}}$ because $X_i \times X_j$ is utility independent of $X_{\overline{ij}}$. Thus, it is not necessary to worry about the value of $x_{\overline{ij}}$ in ascertaining whether the assumption is appropriate.

This corollary indicates that the additive utility function can be considered as a limiting case of the more general multiplicative utility function.

3. ASSESSING THE UTILITY FUNCTION

One important fact about Theorem 1 is that it is operational. All of the assumptions are meaningful to decision makers and it is possible, with a reasonable amount of effort, to ascertain whether they are appropriate for a particular problem. Furthermore, if they do hold, the amount of information needed from the decision maker to specify the utility function is manageable--it only increases linearly with the number of attributes.

Verifying the Assumption

One of the first steps in assessing the utility function is to verify the requisite assumptions. To check whether $X_i \times X_j$ is preferentially indepen-dent of $X_{\overline{ij}}$, we might proceed as follows. First choose an $x_{\overline{ij}}$ with all the components at a relatively undesirable level and find an (x_i, x_j) and (x_i', x_j') such that $(x_i, x_j, x_{\overline{ij}})$ is indifferent to $(x_i', x_j', x_{\overline{ij}})$. Then pick another point $x'_{\overline{ij}}$ with all components at a relatively desirable level and ask the decision maker if $(x_i, x_j, x'_{\overline{ij}})$ is indifferent to $(x_i', x_j', x'_{\overline{ij}})$. This must be true if $X_i \times X_j$ is preferentially independent of $X_{\overline{ij}}$. If the decision maker's answer was affirmative, repeat the same procedure for other pairs of points in the

$X_i \times X_j$ plane with $X_{\overline{ij}}$ fixed at varying levels. If the answers to these questions still indicate preferential independence, then ask if the decision maker is indifferent between $(x_i, x_j, x_{\overline{ij}})$ and $(x_i', x_j', x_{\overline{ij}})$ for any $x_{\overline{ij}}$, does this imply the same indifference would hold for any value of $x_{\overline{ij}}$. A positive answer implies $X_i \times X_j$ is preferentially independent of $X_{\overline{ij}}$.

An obvious way to check whether X_i is utility independent of $X_{\overline{i}}$ is to assess conditional utility functions over X_i given different amounts of $X_{\overline{i}}$. If they are positive linear transformations of each other, the utility independence assumption would be appropriate. More specifically, one could assess certainty equivalents x_i such that $(x_i, x_{\overline{i}})$ is indifferent to a lottery yielding either $(x_i', x_{\overline{i}})$ or $(x_i'', x_{\overline{i}})$ with equal probability. If the certainty equivalent for any lottery did not depend on the amount $x_{\overline{i}}$, then X_i would be utility independent of $X_{\overline{i}}$. In practice, if such a condition held for three or four fifty-fifty lotteries covering the range of X_i for approximately four different values of $x_{\overline{i}}$ covering the range of $X_{\overline{i}}$, one would usually be justified to assume X_i is utility independent of $X_{\overline{i}}$.

The Necessary Information

To use either the additive form (1) or the multiplicative form (2), we need to obtain exactly the same information. To completely specify the n attribute utility function, $u(x_1, x_2, \ldots, x_n)$ we can assess the n single attribute utility functions $u_i(x_i)$ on zero to one scales and the n scaling constants k_i.

Given $\sum\limits_{i=1}^{n} k_i = 1$, then the additive utility function is appropriate. If

$\sum\limits_{i=1}^{n} k_i \neq 1$, the utility function is multiplicative and the additional constant

k in (2) can be found from the k_i values. Thus, as mentioned earlier the additive utility function can be thought of as a special case of the multiplicative utility function.

Suppose we define x_i^o and x_i^* to be the least and most desirable amounts of attribute X_i. Then, the utility function $u_i(x_i)$, which can be assessed using the standard technique discussed in Schlaifer,[15] must be scaled such that

$$u_i(x_i^o) = 0 \tag{3}$$

and

$$u_i(x_i^*) = 1, \tag{4}$$

in order to satisfy the scaling convention of Theorem 1.

Evaluating the Scaling Constants

By solving either (1) or (2) for $u(x_1^o, \ldots, x_{i-1}^o, x_i^*, x_{i+1}^o, \ldots, x_n^o)$, we find

$$u(x_i^*, x_{\bar{i}}^o) \equiv u(x_1^o, \ldots, x_{i-1}^o, x_i^*, x_{i+1}^o, \ldots, x_n^o) = k_i. \tag{5}$$

Thus to assess k_i, we can ask the decision maker for a probability p_i such that he is indifferent between (x_i^*, x_i^o) for certain and a lottery yielding either $\underline{x}^*$ with probability p_i or $\underline{x}^o$ with probability $(1-p_i)$. Since as noted in Theorem 1 we set

$$u(\underline{x}^{0}) = u(x_{1}^{0}, x_{2}^{0}, \ldots, x_{n}^{0}) = 0 \tag{6}$$

and

$$u(\underline{x}^{*}) = u(x_{1}^{*}, x_{2}^{*}, \ldots, x_{n}^{*}) = 1, \tag{7}$$

it follows that

$$u(x_{i}^{*}, x_{\bar{i}}^{0}) = p_{i}, \tag{8}$$

and $k_{i} = p_{i}$. There are more sophisticated ways to assess the scaling constants such as those discussed in Raiffa.[14]

Evaluating Constant k

When the multiplicative form is appropriate, we must determine parameter k. In this case, we can evaulate (2) at $\underline{x}^{*}$ to find

$$1 + k = \prod_{i=1}^{n} (1 + k \, k_{i}). \tag{9}$$

As mentioned earlier, if $\displaystyle\sum_{i=1}^{n} k_{i} = 1$, the utility function is additive.

If $\displaystyle\sum_{i=1}^{n} k_{i} > 1$, then using (2) and (9), it can be shown that the properties of

utility functions can only be preserved given that $-1 < k < 0$. In this case, by interatively evaluating (9) given the k_{i}, one can converge to the appropriate value of k, call it k_{t}. First set $k = k_{s}$ and substitute this into (9). If the right hand side is smaller than the left hand side, then $k_{t} < k_{s}$. If the r.h.s. is greater than the l.h.s., then $k_{t} > k_{s}$.

When $\sum\limits_{i=1}^{n} k_i < 1$, it follows that from similar reasoning that $k_t > 0$.

Let us arbitrarily set $k = k_s$ in (9). If the r.h.s. > l.h.s., then $k_t > k_s$, whereas if the l.h.s. > r.h.s., then $k_t < k_s$.

4. RELATED RESULTS

Many others have worked on problems related to obtaining multi-attributed utility functions. In this section, we will briefly summarize some of the results closely related to those discussed here. A more comprehensive survey is found in Fishburn.[6]

In a series of papers, Fishburn[3,4,5] has derived necessary and sufficient conditions for multiattributed utility functions to be additive. His conditions require that the desirability of any lottery over X only depend on the marginal probability distributions over the X_i and not on the joint probability distribution.

Pollak[13] and Meyer[12] both derive necessary and sufficient conditions for an n attribute utility function to be additive or multiplicative. Thus, their assumptions are equivalent to the conditions specified in Theorem 1. However, their assumptions are much stronger than ours, and because they concern utility independence conditions with several attributes varying simultaneously, they are much more difficult to verify. Since these results are directly related to mine, let us state them.

THEOREM 2. (Pollak). If $X_{\bar{i}}$ is utility independent of X_i, for all i, i $\geq$ 2, the utility function $u(\underline{x})$ is additive or log additive (i.e., multiplicative).

To be precise, Pollak does not assume utility independence conditions, but rather he assumes his "weak independence axiom" which is equivalent to the conditions stated. This axiom states an individual's preferences between any fifty-fifty lottery yielding either $(x_i, x'\frac{}{i})$ or $(x_i, x''\frac{}{i})$ or a consequence $(x_i, x'''\frac{}{i})$ should be independent of the amount x_i for all lotteries, for all $x'''\frac{}{i}$, and for all choices of i.

To prove Meyer's result, we need to define

$$X^m \equiv X_m \times X_{m+1} \times \dots \times X_n, \ m \leq n \text{ and } {}^m X \equiv X_1 \times X_2 \times \dots \times X_m.$$

THEOREM 3. (Meyer). If X^m is utility independent of ${}^{m-1}X$, for all $m = 2, 3, \dots, n$ and if ${}^{n-1}X$ is utility independent of X_n, then u(x) is either multiplicative or additive.

Proofs of these theorems are found in their respective references. Note that both require n assumptions, as does Theorem 1. However, both sets of assumptions in this section require the decision maker to express cardinal preferences (i.e., preferences over lotteries) with from two to n-1 attributes varying at the same time. The conditions of Theorem 1 require only ordinal preferences (i.e., preferences over consequences) with two attributes varying and one set of cardinal preferences over one attribute.

A number of related results which follows from various sets of utility independence assumptions are found in Raiffa[14] and Keeney.[8,9,10] Papers considering the implications of preferential independence include Debreu,[1] Gorman,[7] Raiffa,[14] and Ting.[16]

5. AN EXAMPLE

Recently, the results of this paper were used on a study to select a strategy for developing the airport facilities of Mexico City for the rest of the century. Details of the overall study can be found in de Neufville and Keeney.[2] A part of the study involved assessing a utility function over six attributes useful for indicating the effectiveness of various developmental strategies. The attributes were

X_1 ≡ total cost in millions of pesos;

X_2 ≡ the practical capacity in terms of the number of aircraft operations per hour;

X_3 ≡ access time to and from the airport in minutes;

X_4 ≡ the number of people seriously injured or killed per aircraft accident;

X_5 ≡ the number of people displaced by airport development; and

X_6 ≡ the number of people subjected to a high noise level (measured by an index combining decibel level and frequency of occurrences).

Rather than repeat specifics of the assessment procedure found elsewhere,[11] let me say that the conditions of Theorem 1 were verified with the Director of the Department of Airports in the Ministry of Public Works, as well as independently with his assistants. The Department of Airports has the responsibility for building and maintaining all airports in Mexico. Then the conditional utility functions $u_i(x_i)$ and the scaling constants k_i were assessed. We found that the utility function was multiplicative. This utility function was

used for evaluating the effectiveness of the proposed strategies for developing the Mexico City airport facilities.

6. CONCLUSIONS

This paper presents a set of necessary and sufficient assumptions to imply that a multiattributed utility function is either multiplicative or additive. It is shown that the additive utility function can be considered as a limiting case of the multiplicative utility function. Our assumptions are operational-- that is, it is reasonable to expect that by questioning a decision maker, one would be able to ascertain whether or not the assumptions hold for his particular problem. The questions which would be appropriate to verify the assumptions are discussed. Finally, an example is mentioned where the main result was used to structure a utility function over six attributes in a real-world problem.

7. ACKNOWLEDGMENT

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Appendix

In this appendix, details of the proofs of the three lemmas stated in Section 2 are given.

The notation used in proving the first two lemmas will be altered slightly from that in the main text in order to avoid subscripts where necessary and to simplify expressions. We will use $S, T, Y,$ and Z as attributes rather than $X_i, X_j, X_{\overline{ij}}$, etc. Thus, for instance s will be a specific amount of S. Furthermore, since we will always use the utility function $u(\underline{x})$, which will now be written $u(s, t, \ldots, z)$, when an attribute is at its least desirable amount, designated as s^o for example, we may delete it in the function when no ambiguity will result. Thus, rather than write $u(s^o, t^o, y, z)$, $u(s^o, t^o, y^o, z)$, and $b(t^o, z)$, we will use $u(y, x)$, $u(z)$, and $b(z)$.

Lemma 1. If X_i is utility independent of $X_{\overline{i}}$ and if

$X_i \times X_j$ is preferentially independent of

$X_{\overline{ij}}$, then $X_i \times X_j$ is utility independent

of $X_{\overline{ij}}$.

Proof. Let $S \equiv X_i$, $T = X_j$, and $Y \equiv X_{\overline{ij}}$. Therefore $X_{\overline{i}} = X_j \times X_{\overline{ij}} = T \times Y$. We can mathematically represent the condition that S is utility independent of $T \times Y$ as

$$u(s, t, y) = u(t, y) + b(t, y) u(s). \tag{A-1}$$

Also, since S x T is preferentially independent of Y, we know

$$u(s,t,y^o) = u(s^+,t^+,y^o) \longrightarrow u(s,t,y) = u(s^+,t^+,y), \; \forall \, y. \qquad \text{(A-2)}$$

Let us choose s' such that

$$u(s,t,y^o) = u(s',t^o,y^o). \qquad \text{(A-3)}$$

Then, substituting (A-1) into (A-3) yields

$$u(t) + b(t)\, u(s) = u(s') \qquad \text{(A-4)}$$

since setting $t = t^o$ and $y = y^o$ in (A-1) indicates

$$b(t^o, y^o) = 1. \qquad \text{(A-5)}$$

From (A-3) and the preferential independence assumption,

$$u(s,t,y) = u(s',t^o,y), \; \forall \, y. \qquad \text{(A-6)}$$

Evaluating both sides of (A-6) with (A-1) and combining the results with (A-4), we find

$$u(t,y) + b(t,y)\, u(s) = u(y) + b(y)\, [u(t) + b(t)\, u(s)]. \qquad \text{(A-7)}$$

If we set $s = s^o$ in (A-7),

$$u(t,y) = u(y) + b(y)\, u(t), \qquad \text{(A-8)}$$

which can be substituted back into (A-7) to yield

$$b(t,y) = b(t)\, b(y). \qquad \text{(A-9)}$$

Now, substituting (A-8) and (A-9) into (A-1) gives us

$$
\begin{aligned}
u(s,t,y) &= u(y) + b(y)\, u(t) + b(t)\, b(y)\, u(s) \\
&= u(y) + b(y)\, [u(t) + b(t)\, u(s)] \\
&= u(y) + b(y)\, u(s,t). \qquad \text{(A-10)}
\end{aligned}
$$

Equation (A-10) says S x T is utility independent of Y which is the desired result.

Lemma 2 is now proven for $n \geq 4$, since it does not apply in the case when $n = 3$.

Lemma 2. If $X_i \times X_j$ is utility independent of $X_{\overline{ij}}$ and $X_i \times X_k$ are utility independent of $X_{\overline{ik}}$, then $X_i \times X_j \times X_k$ is utility independent of $X_{\overline{ijk}}$.

<u>Proof of Lemma 2.</u> $(n \geq 4)$. Let $S \equiv X_i$, $T \equiv X_j$, $Y \equiv X_k$, and $Z \equiv X_{\overline{ijk}}$. The given utility independence conditions imply

$$u(s,t,y,z) = u(y,z) + a(y,z)\, u(s,t) \tag{A-11}$$

and

$$u(s,t,y,z) = u(t,z) + b(t,z)\, u(s,y). \tag{A-12}$$

Substituting (A-11) into (A-12) and then (A-12) into (A-11) gives us

$$u(s,t,y,z) = u(z) + b(z)\, u(y) + a(y,z)\, [u(t) + b(t)\, u(s)] \tag{A-13}$$

and

$$u(s,t,y,z) = u(z) + a(z)\, u(t) + b(t,z)\, [u(y) + a(y)\, u(s)]$$

which can be equated with $t = t^o$ to yield

$$a(y,z) = b(z)\, a(y). \tag{A-14}$$

Then, substituting (A-14) back into (A-13) gives

$$u(s,t,y,z) = u(z) + b(z)\, [u(y) + a(y)\, u(s,t)]$$
$$= u(z) + b(z)\, u(s,t,y) \tag{A-15}$$

which proves Lemma 2.

Lemma 3. If $X_{\bar{i}}$ is utility independent of X_i for

n-1 of the i = 1, 2, ..., n, and n $\geq$ 3,

then either (1) or (2) is valid.

Proof. With no loss of generality, we will assume $X_{\bar{i}}$ is utility independent of X_i for i = 1, 2, ..., n-1 which implies

$$u(\underline{x}) = u(x_i) + c_i(x_i) u(x_{\bar{i}}), \quad i = 1, 2, \ldots, n-1. \tag{A-16}$$

Setting all $x_i = x_i^o$ except x_1 and x_j, j = 2, 3, ..., n-1, we get the equality

$$u(x_1, x_j) = u(x_1) + c_1(x_1) u(x_j) = u(x_j) + c_j(x_j) u(x_1)$$

or

$$\frac{c_j(x_j) - 1}{u(x_j)} = \frac{c_1(x_1) - 1}{u(x_1)} = k, \quad j = 2, 3, \ldots, n-1. \tag{A-17}$$

Thus, it follows that

$$c_i(x_i) = k \, u(x_i) + 1, \quad \text{for all } i = 1, 2, \ldots, n-1. \tag{A-18}$$

We can repeatedly use (A-16) to obtain

$$u(\underline{x}) = u(x_1) + c_1(x_1) u(x_2, x_3, \ldots, x_n)$$

$$= u(x_1) + c_1(x_1) [u(x_2) + c_2(x_2) u(x_3, x_4, \ldots, x_n)]$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$= u(x_1) + c_1(x_1) u(x_2) + c_1(x_1) c_2(x_2) u(x_3) + \ldots + \tag{A-19}$$
$$c_1(x_1) \ldots c_{n-1}(x_{n-1}) u(x_n).$$

Substituting (A-18) into (A-19) yields

$$u(\underline{x}) = u(x_1) + [ku(x_1) + 1] u(x_2) + [ku(x_1) + 1][ku(x_2) + 1] u(x_3) + \ldots$$

$$+ [ku(x_1) + 1][ku(x_2) + 1] \ldots [ku(x_{n-1}) + 1] u(x_n). \tag{A-20}$$

When $k = 0$, (A-16) becomes the additive utility function

$$u(\underline{x}) = \sum_{i=1}^{n} u(x_i). \tag{A-21}$$

When $k \neq 0$, we can multiply both sides of (A-20) by k, then add 1 to each, and rearrange terms to find

$$ku(\underline{x}) + 1 = \prod_{i=1}^{n} [ku(x_i) + 1]. \tag{A-22}$$

Recall that $u(x_i)$ actually means $u(x_1^o, x_2^o, \ldots, x_i, x_{i+1}^o, \ldots, x_n^o)$. Since we define $u(x_i) = k_i u_i(x_i)$ so the $u_i(x_i)$ can be scaled from zero to one, (A-21) and (A-22) become respectively

$$u(\underline{x}) = \sum_{i=1}^{n} k_i u_i(x_i) \tag{1}$$

and

$$ku(\underline{x}) + 1 = \prod_{i=1}^{n} [k k_i u_i(x_i) + 1], \tag{2}$$

which proves Lemma 3.

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